

Bericht

On the Logic of Quantum Mechanics

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Istituto Nazionale di Fisica Nucleare — Sez. Genova(Z. Naturforsch. **28 a**, 1516—1530 [1973] ; received 9 December 1972)*To Konrad Bleuler on the occasion of his 60th birthday*

We are concerned with the formulation of the essential features of quantum theory in an abstract way, utilizing the mathematical language of proposition lattice theory. We review this approach giving a set of consistent axioms which enables to achieve the relevant results: the formulation and the essential role of the superposition principle is particularly examined.

1. Introduction

The existence of a quantum logic and its algebraic formulation^{1,2} has received increasing interest and significant developments in the last years, mainly after the relevant contributions of Mackey³. The approach we are concerned with is the one going through the study of the lattice structure of yes-no experiments: the so called propositional calculus, which has been highly developed by Piron⁴, Jauch⁵, Varadarajan⁶. Among other approaches we recall the Jordan algebras⁷, the Segal method^{8,9} the C*-algebra axiomatization¹⁰, the Ludwig studies^{11,12}; for connections see, e.g., the references^{13,14,15,16}.

A large number of papers appeared on the lattice approach and a variety of partially overlapping axiom sets have been advanced. On account of these contributions we try to review the subject with some economy in the choice of the axioms needed to achieve the more relevant results. In particular we focus attention and give some new results on the role of the superposition principle which we adopt in the form proposed by Varadarajan⁶.

2. States of a System, Propositions

The so called propositional calculus of quantum mechanics tacitly assumes the possibility of separating a portion of the universe — to be called a physical system — whose interaction with the rest of the universe might be described through the notion of

- a) preparation of the system, i.e., the set of the procedures used to separate and prepare it: these procedures actually define the system;

- b) observable on the system, i.e., the observation of a macroscopic measurement device interacting, in a reproducible way, with the system.

According to this, the preparation of the system is, by definition, modified after an observation on it is made.

As well known from information theory, one is always allowed to restrict to observables having dichotomic output: without loss of generality we can assume the output to be realized by the appearance of the marks “yes”, “no” on the panel of the measurement device. Observables with such a dichotomic output are referred to as propositions. Starting with any observable, it is clear how to modify the apparatus in order to get propositions: put a window on the scale of the measurement instrument — in order to separate a portion of it — and act the output “yes” if the scaler appears within the window, “no” if not; of course any observable looks equivalent, from the information standpoint, to a suitable set of propositions.

The set of all propositions on a system will be denoted by Q , single propositions by the letters p, q, r, \dots .

Propositions are assumed to resolve any output uncertainty into one of the marks “yes” or “no”: i.e., one and only one answer appears. Thus, the information carried by a proposition p is entirely specified by the probability $\psi(p, \xi; \text{yes})$ of the “yes” output when the preparation of the system is ξ ; the probability of the “no” output being $\psi(p, \xi; \text{no}) = 1 - \psi(p, \xi; \text{yes})$.

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Two preparations ξ, ξ' are equivalent if

$$\psi(p, \xi; \text{yes}) = \psi(p, \xi'; \text{yes}) \quad \forall p \in Q;$$

this equivalence relation splits the set of all preparations into equivalence classes which will be called the states of the system. The set of the states will be denoted by S , single states by the letters α, β, \dots .

If ξ belongs to the state α , let us put

$$m_\alpha(p) = \psi(p, \xi; \text{yes}), \quad \xi \in \alpha. \quad (2.1)$$

By definition we get the implication

$$m_\alpha(p) = m_\beta(p) \quad \forall p \in Q \Rightarrow \alpha = \beta; \quad (2.2)$$

moreover, we shall assume that distinct propositions have to assign different probabilities of the yes output for one state at least, i.e.:

$$m_\alpha(p) = m_\alpha(q) \quad \forall \alpha \in S \Rightarrow p = q. \quad (2.3)$$

The only functions defined on the marks "yes", "no" and having values in the same marks are

$$\begin{aligned} f_0: f_0(\text{yes}) &= \text{no}, & f_0(\text{no}) &= \text{no}, \\ f_1: f_1(\text{yes}) &= \text{yes}, & f_1(\text{no}) &= \text{yes}, \\ f_2: f_2(\text{yes}) &= \text{no}, & f_2(\text{no}) &= \text{yes}, \end{aligned}$$

besides the identical function. Acting with the function f_0 (respectively with f_1) on the output panel of a proposition p , one gets an apparatus whose answers are both labeled by "no" (respectively by "yes"): this defines a proposition to be denoted by 0_Q (respectively by 1_Q) whose output is always and certainly "no" (respectively "yes"). Acting with the function f_2 one gets a proposition which is derived from p by the simple interchanging of the "yes", "no" marks: it will be called the canonical complement of p and denoted by p^\perp . Obviously $(p^\perp)^\perp = p$. The definitions of $0_Q, 1_Q, p^\perp$ are equivalent to state

$$m_\alpha(0_Q) = 0 \quad \forall \alpha \in S, \quad (2.4)$$

$$m_\alpha(1_Q) = 1 \quad \forall \alpha \in S, \quad (2.5)$$

$$m_\alpha(p^\perp) = 1 - m_\alpha(p) \quad \forall \alpha \in S, \quad (2.6)$$

3. Mixtures, Pure States, Superpositions

A state $\alpha \in S$ is said to be a mixture of the states $\alpha_1, \alpha_2, \dots, \alpha_i, \dots$ with weights $t_1, t_2, \dots, t_i, \dots$ ($0 \leq t_i \leq 1$), if

$$m_\alpha(p) = \sum_i t_i m_{\alpha_i}(p), \quad \forall p \in Q. \quad (3.1)$$

The state α is uniquely determined (see (2.2)) by the sequences $\{\alpha_i\}, \{t_i\}$ and will be denoted by

$$\alpha = \sum_i t_i \alpha_i. \quad (3.2)$$

Of course the weights satisfy the relation $\sum_i t_i = 1$, as one sees putting $p = 1_Q$ into (3.1).

A state which cannot be expressed as a mixture of other states is said to be a pure state. The set of all pure states will be denoted by P . We now adopt the following

Axiom 1.

- (i) P is a non-empty set;
- (ii) any non pure state may be written as a mixture of pure states;
- (iii) for any given sequence of states and weights there exists the corresponding mixture.

This axiom gives to S the structure of the closed convex hull of its extreme points, which are the elements of P .

Without loss of generality the mixture (3.2) may be put in the form

$$\alpha = t\beta + (1-t)\gamma, \quad 0 \leq t \leq 1, \quad \beta, \gamma \in S, \quad (3.2')$$

as it may be seen letting

$$t = t_1, \quad \beta = \alpha_1, \quad \gamma = \sum_{j \geq 2} t_j / (1 - t_1) \alpha_j;$$

the existence of the mixture γ is guaranteed by the statement (iii) of the Axiom 1.

In classical mechanics the states represented by a point in the phase space \mathcal{I} of the system are its pure states: as an example of classical mixture we could refer to the state of a molecule coming out through a small hole in the box containing a gas.

We have now to provide a definition of superposition of states, able to meet the corresponding quantum mechanical notion. Given any non-empty subset \mathcal{P} of S we write

$$\mathcal{P}(q) \cong \lambda$$

to mean that $m_\alpha(q) = \lambda$ for all $\alpha \in \mathcal{P}$. According to Varadarajan⁶ we give the following definition. Given any non-empty subset \mathcal{P} of P , a state β not belonging to \mathcal{P} is said to be a superposition of the states belonging to \mathcal{P} if any proposition $q \neq 0_Q$ such that $\mathcal{P}(q) \cong 0$ makes true the implication

$$\mathcal{P}(q) \cong 0 \Rightarrow m_\beta(q) = 0 \quad (q \neq 0_Q).$$

We further define the closure $\bar{\mathcal{P}}$ of \mathcal{P} as the set of the states belonging to \mathcal{P} and of all pure states which are superpositions of the elements of \mathcal{P} . Hence

$$\bar{\mathcal{P}} \supseteq \mathcal{P}. \quad (3.3)$$

If $\bar{\mathcal{P}} = \mathcal{P}$, the subset \mathcal{P} is said to be closed. We shall denote by M the set of all subsets of P and

by \mathcal{M} the set of all closed subsets of P , i.e.:

$$\mathcal{M} = \{\mathcal{P} \subseteq P : \mathcal{P} = \overline{\mathcal{P}}\}. \quad (3.4)$$

Clearly, any mixture of states is also a superposition. In classical mechanics the converse too is true: any superposition is a mixture, i.e., any subset of P turns out to be closed, so that $\mathcal{M} = M$. Things are different in quantum mechanics where we know that superposing states one can get pure states, hence a subset of P is, in general, not closed and \mathcal{M} is strictly contained in M ; the subset of P containing two linear polarization states of an electromagnetic wave is an example of a non closed subset, since the various elliptical polarization states, which are superpositions of the two linear polarization states, also belong to its closure.

The following two lemmas give a number of properties of the subset of P^{17} .

Lemma (3.a). For every non empty subsets \mathcal{P} of P the following is true:

- (i) $\mathcal{P}(q) \cong 0$ if and only if $\overline{\mathcal{P}}(q) \cong 0$, ($q \neq 0_Q$);
- (ii) $\overline{\overline{\mathcal{P}}} = \overline{\mathcal{P}}$;
- (iii) if $\mathcal{P}' \subset P$ is such that $\overline{\mathcal{P}'} = \mathcal{P}'$ and $\mathcal{P}' \supseteq \mathcal{P}$ then $\mathcal{P}' \supseteq \overline{\mathcal{P}}$;
- (iv) if $\mathcal{P}' \subset P$ is such that $\mathcal{P}' \supseteq \mathcal{P}$ then $\overline{\mathcal{P}'} \supseteq \overline{\mathcal{P}}$.

Lemma (3.b). For every set $\{\mathcal{P}_i\}$ of subset of P , where i runs over an index set I , the intersections $\bigcap_{i \in I} \mathcal{P}_i$, $\bigcap_{i \in I} \overline{\mathcal{P}_i}$ of all the sets belonging to the sequences $\{\mathcal{P}_i\}$, $\{\overline{\mathcal{P}_i}\}$ fulfill the relations

$$(i) \bigcap_{i \in I} \overline{\mathcal{P}_i} = \overline{\bigcap_{i \in I} \mathcal{P}_i}; \quad (ii) \bigcap_{i \in I} \overline{\mathcal{P}_i} \supseteq \overline{\bigcap_{i \in I} \mathcal{P}_i}.$$

Let us now recall a few definitions. An ordering relation in a set E is any reflexive, transitive, antisymmetric relation in E : writing $x \leq y$ to mean that the pair (x, y) of elements of E belongs to the ordering relation, one demands

$$\begin{aligned} x &\leq x, \quad \forall x \in E; \\ x &\leq y \quad \text{and} \quad y \leq z \Rightarrow x \leq z; \\ x &\leq y \quad \text{and} \quad y \leq x \Rightarrow x = y. \end{aligned} \quad (3.5)$$

If E is partially ordered there is at most one element called zero and denoted by 0_E such that $0_E \leq x$, $\forall x \in E$; similarly there is at most one unit element denoted by 1_E such that $x \leq 1_E$, $\forall x \in E$. For any non empty subset \mathcal{E} of E there exists at most one element c of E such that (i) $x \leq c$, $\forall x \in \mathcal{E}$, (ii) if $d \in E$ is such that $x \leq d$, $\forall x \in \mathcal{E}$, then $c \leq d$.

We shall write $\bigvee_{x \in \mathcal{E}} x$ for c whenever it exists and call

it the least upper bound of the elements of \mathcal{E} . Similarly there exists at most one element b of E such that (i) $b \leq x$, $\forall x \in \mathcal{E}$, (ii) if $d \in E$ is such that $d \leq x$, $\forall x \in \mathcal{E}$, then $d \leq b$. We shall write $\bigwedge_{x \in I} x$ for

b whenever it exists and call it the greatest lower bound of the elements of \mathcal{E} . If the elements of \mathcal{E} are labeled by an index i running over an index set I , we shall prefer the obvious notations $\bigvee_{x \in I} x_i$ and

$\bigwedge_{x \in I} x_i$; if \mathcal{E} is finite, say $\mathcal{E} = \{x_1, \dots, x_n\}$ it is customary to write $x_1 \vee x_2 \vee \dots \vee x_n$ and $x_1 \wedge x_2 \wedge \dots \wedge x_n$ in place of $\bigvee_{x \in \mathcal{E}} x$ and $\bigwedge_{x \in \mathcal{E}} x$ respectively.

The partially ordered set E is said to be a complete lattice if (i) the elements 0_E , 1_E exist, (ii) the least upper bound and the greatest lower bound exist for any subset.

Now we return to the set \mathcal{M} defined by (3.4) and prove the

Theorem (3.c). \mathcal{M} is a complete lattice.

Since the elements of \mathcal{M} are sets themselves, the usual inclusion relation (\subseteq) among sets clearly defines an ordering relation in \mathcal{M} ; moreover $0_{\mathcal{M}} \neq \emptyset$, $1_{\mathcal{M}} = P$. Consider any subset $\{\mathcal{P}_i\}$ of \mathcal{M} where i runs over I . By definition $\mathcal{P}_i = \overline{\mathcal{P}_i}$, $\forall i \in I$; hence the Lemma (3, b) guarantees that the set intersection $\bigcap_{i \in I} \mathcal{P}_i$ is closed and therefore belongs to \mathcal{M} .

Moreover if $\mathcal{P}_0 \in \mathcal{M}$ is such that $\mathcal{P}_0 \subseteq \mathcal{P}_i$, $\forall i \in I$, then $\mathcal{P}_0 \subseteq \bigcap_{i \in I} \mathcal{P}_i$: this shows that the intersection

$\bigcap_{i \in I} \mathcal{P}_i$ is the greatest lower bound $\bigwedge_{i \in I} \mathcal{P}_i$. Consider now the set union $\bigcup_{i \in I} \mathcal{P}_i$ of the elements of $\{\mathcal{P}_i\}$. Its closure $\overline{\bigcup_{i \in I} \mathcal{P}_i}$ is closed by virtue of the statement (ii)

of Lemma (3, a) and therefore belongs to \mathcal{M} ; moreover if $\mathcal{P}_0 \in \mathcal{M}$ is such that $\mathcal{P}_0 \supseteq \mathcal{P}_i$, $\forall i \in I$ then $\mathcal{P}_0 \supseteq \bigcup_{i \in I} \mathcal{P}_i$ and the statement (iii) of Lemma (3, a)

yields $\mathcal{P}_0 \supseteq \overline{\bigcup_{i \in I} \mathcal{P}_i}$. Hence $\bigcup_{i \in I} \mathcal{P}_i$ is the least upper bound $\bigvee_{i \in I} \mathcal{P}_i$. This completes the proof.

Clearly, also the set M of all subsets of P is a complete lattice. In fact the usual inclusion relation (\subseteq) among sets gives to M the structure of a partially ordered set, with $0_M = \emptyset$, $1_M = P$; moreover the greatest lower bound and the least upper bound of any set of elements of M is clearly given by the

corresponding set intersection an union. The Boolean character of the lattice M is also implied.

It must be stressed that the difference in the lattice structure of \mathcal{M} and M lies in the fact that the least upper bound in \mathcal{M} is not just the set union but the closure of this union. Hence \mathcal{M} is not in general a Boolean lattice and is not a sublattice of M since the rule giving the least upper bound in \mathcal{M} is not the restriction to \mathcal{M} of the rule operating in M .

As already remarked, the phase space \mathcal{I} of a classical system coincides with the set P of the pure states, moreover the set Q of propositions may be represented as the set of all subsets of \mathcal{I} . Hence Q is isomorphic to the Boolean lattice M .

In the next sections it will be seen that for quantum mechanical systems Q is made isomorphic to the lattice \mathcal{M} .

4. Partial Ordering and Orthocomplementation in the Set of Propositions

Making use of Axiom 1 we can write the implication (2.3) in an equivalent form by restricting the states α to pure states:

$$m_\alpha(p) = m_\alpha(q), \quad \forall \alpha \in P \Rightarrow p = q; \quad (4.1)$$

i.e., the set P is sufficient to recognize different propositions.

Recalling the requirements (3.5) which define an ordering relation and looking at (4.1) it is straightforward to check that the functions m_α define a partial ordering in the set Q of propositions through the rule

$$p \stackrel{w}{\leq} q \Leftrightarrow m_\alpha(p) \leq m_\alpha(q), \quad \forall \alpha \in P. \quad (4.2)$$

This ordering relation is not the only one which looks physically reasonable: we shall refer to it as the weak ordering in Q .

In fact, another ordering relation can be introduced, which rests on the logical implication of two propositions: p might be defined less than q if q surely gives the answer "yes" whenever p does. To put this in a definite form, introduce the set of pure states

$$\mathcal{P}_1(p) = \{\alpha \in P : m_\alpha(p) = 1\} \quad (4.3)$$

and assume the rule

$$p \stackrel{s}{\leq} q \Leftrightarrow \mathcal{P}_1(p) \subseteq \mathcal{P}_1(q); \quad (4.4)$$

it is obviously a reflexive and transitive relation,

but to meet the last requirement (3.5) we have to adopt the next

Axiom 2.

$$\mathcal{P}_1(p) = \mathcal{P}_1(q) \Rightarrow p = q.$$

This axiom is stronger than (4.1), in fact it states that a suitable subset of P is sufficient to recognize different propositions. Hence (4.4) defines an ordering relation; we shall refer to it as the strong ordering in Q . The name is motivated by the obvious implication

$$p \stackrel{w}{\leq} q \Rightarrow p \stackrel{s}{\leq} q,$$

which shows that the relation $\stackrel{s}{\leq}$ orders, in principle, more propositions than $\stackrel{w}{\leq}$ does.

We now remark that the propositions 0_Q , 1_Q defined by (2.4), (2.5) are such that, for every $p \in Q$:

$$0_Q \stackrel{w}{\leq} p, \quad 0_Q \stackrel{s}{\leq} p, \quad p \stackrel{w}{\leq} 1_Q, \quad p \stackrel{s}{\leq} 1_Q,$$

where we noticed that $\mathcal{P}_1(1_Q) = P$ while $\mathcal{P}_1(0_Q) = \emptyset$. Thus the propositions 0_Q , 1_Q behave as zero and unit elements with respect to both orderings. Since the zero element of a partially ordered set is unique, we may state, referring to (4.4):

$$\mathcal{P}_1(p) \neq \emptyset \quad \text{if and only if} \quad p \neq 0_Q, \quad (4.5)$$

for $\mathcal{P}_1(p) = \emptyset$ implies, through Axiom 2, $p = 0_Q$, and, conversely, $p = 0_Q$ obviously implies $\mathcal{P}_1(p) = \emptyset$.

Let us now remind a relevant definition. Let E be a partially ordered set with zero and unit elements 0_E , 1_E : a one to one mapping $x \mapsto x'$ of E onto itself is said to be an orthocomplementation in E if

$$(i) \quad (x')' = x, \quad \forall x \in E;$$

$$(ii) \quad x \vee x' = 1_E \quad \text{and} \quad x \wedge x' = 0_E, \quad \forall x \in E; \quad (4.6)$$

$$(iii) \quad x \leq y \Leftrightarrow y' \leq x'.$$

If such a mapping $x \mapsto x'$ exists, we shall refer to E as an orthocomplemented set. Clearly $0'_E = 1_E$.

The following equalities hold, if \mathcal{E} is any non empty subset of E :

$$\bigwedge_{x \in \mathcal{E}} x' = (\bigvee_{x \in \mathcal{E}} x)', \quad (4.7)$$

$$\bigvee_{x \in \mathcal{E}} x' = (\bigwedge_{x \in \mathcal{E}} x)'. \quad (4.8)$$

In fact, for any $y \in \mathcal{E}$, we get $y \leq \bigvee_{x \in \mathcal{E}} x$, hence $(\bigvee_{x \in \mathcal{E}} x)' \leq y'$, so that $(\bigvee_{x \in \mathcal{E}} x)' \leq \bigwedge_{y \in \mathcal{E}} y'$; moreover, for

any $x' \in \mathcal{E}$, we get $\bigwedge y' \leq x'$, hence $x \leq (\bigwedge y')'$, so that $\bigvee_{x \in \mathcal{E}} x \leq (\bigwedge_{y \in \mathcal{E}} y')'$ or $(\bigvee_{x \in \mathcal{E}} x)' \geq \bigwedge_{y \in \mathcal{E}} y'$: this proves (4.7). One similarly proves (4.8).

Consider now the set of propositions Q , equipped with the weak or the strong ordering; when dealing with the least upper bounds or the greatest lower bounds we shall need to specify (by the appropriate superscription) the ordering we are referring to. Let us examine whether the one to one mapping $p \mapsto p^\perp$, which transforms a proposition p into its canonical complement [see (2.6)], is an orthocomplementation with respect to the orderings (4.2), (4.4). The first requirement (4.6) reads $(p^\perp)^\perp = p$, $\forall p \in Q$ and is obviously met, no matter what the ordering is. Consider first the weak ordering. The requirement (iii) reads $m_\alpha(p) \leq m_\alpha(q) \Leftrightarrow m_\alpha(q^\perp) \leq m_\alpha(p^\perp)$, $\forall \alpha \in P$ and is met due to the definition (2.6); noticing that $m_\alpha(p \overset{w}{\vee} p^\perp) \geq m_\alpha(p)$, and $m_\alpha(p \overset{w}{\vee} p^\perp) \geq 1 - m_\alpha(p)$, $\forall \alpha \in P$, we get $\mathcal{P}_0(p \overset{w}{\vee} p^\perp) = \emptyset$, hence from (4.5) it follows $p \overset{w}{\vee} p^\perp = 1_Q$ so that the requirement (ii) is obtained too. Consider now the strong ordering. The last requirement (4.6) reads

$$\mathcal{P}_1(p) \subseteq \mathcal{P}_1(q) \Leftrightarrow \mathcal{P}_1(q^\perp) \subseteq \mathcal{P}_1(p^\perp)$$

and has to be imposed as a new axiom; introducing the notation

$$\mathcal{P}_0(p) = \{\alpha \in P : m_\alpha(p) = 0\} \quad (4.9)$$

and remarking that $\mathcal{P}_1(p^\perp) = \mathcal{P}_0(p)$ we adopt the following

Axiom 3.

$$\mathcal{P}_1(p) \subseteq \mathcal{P}_1(q) \Leftrightarrow \mathcal{P}_0(q) \subseteq \mathcal{P}_0(p).$$

We have still to check the second requirement (4.6). By definition of least upper bound we get $\mathcal{P}_1(p \overset{s}{\vee} p^\perp) \supseteq \mathcal{P}_1(p)$ and $\mathcal{P}_1(p \overset{s}{\vee} p^\perp) \supseteq \mathcal{P}_1(p^\perp) = \mathcal{P}_0(p)$; by virtue of Axiom 3 we write the first inequality in the form $\mathcal{P}_0(p) \supseteq \mathcal{P}_0(p \overset{s}{\vee} p^\perp)$ and the comparison with $\mathcal{P}_1(p \overset{s}{\vee} p^\perp) \supseteq \mathcal{P}_0(p)$ leads to $\mathcal{P}_1(p \overset{s}{\vee} p^\perp) \supseteq \mathcal{P}_0(p \overset{s}{\vee} p^\perp)$ which implies $\mathcal{P}_0(p \overset{s}{\vee} p^\perp) = \emptyset$, i.e., $\mathcal{P}_1((p \overset{s}{\vee} p^\perp)^\perp) = \emptyset$. With reference to (4.5) it follows $(p \overset{s}{\vee} p^\perp)^\perp = 0_Q$, i.e., $p \overset{s}{\vee} p^\perp = 1_Q$ as required.

To summarize, Axiom 2 is seen to guarantee that the mapping $p \mapsto p^\perp$ is an orthocomplementation with respect to the weak ordering; Axiom 3 has to

be adopted too — as we shall do — in order to guarantee that $p \mapsto p^\perp$ is an orthocomplementation with respect to the strong ordering.

In this Section we have discussed the partial ordering and the orthocomplementation in Q making use of pure states only. However, on account of Axiom 1, it makes no difference to remove throughout the limitation to pure states. The proof of this is straightforward and rests on the implications [see (3.1), (3.2)]

$$m_{\Sigma_i \alpha_i}(p) = 1 \Leftrightarrow m_{\alpha_j}(p) = 1, \quad j = 1, 2, \dots, \quad (4.10)$$

$$m_{\Sigma_i \alpha_i}(p) = 0 \Leftrightarrow m_{\alpha_j}(p) = 0, \quad j = 1, 2, \dots \quad (4.11)$$

The properties stated by Axioms 2 and 3 are easily proved within the scheme of classical mechanics. In fact, as already reminded, there is a one to one correspondence, to be denoted by $p \mapsto \mathcal{V}_p$, between propositions and the subsets of $P = \mathcal{J}$; moreover any pure state represented in \mathcal{J} by a point of \mathcal{V}_p belongs to $\mathcal{P}_1(p)$, while any pure state represented by a point of the set complement $\mathcal{J} - \mathcal{V}_p = \mathcal{V}_{p^\perp}$ belongs to $\mathcal{P}_0(p) = \mathcal{P}_1(p^\perp)$. Hence it is clear that

$p \overset{s}{\leq} q$ means $\mathcal{V}_p \subseteq \mathcal{V}_q$ and this implies $m_\alpha(p) \leq m_\alpha(q)$, $\forall \alpha \in P$. Since $p \overset{s}{\leq} q \Rightarrow p \overset{w}{\leq} q$ we may conclude that for classical mechanics the strong ordering and the weak one are the same thing. $p \vee q$ and $p \wedge q$ are obtained by the usual set union and intersection $\mathcal{V}_p \cup \mathcal{V}_q$ and $\mathcal{V}_p \cap \mathcal{V}_q$; therefore the distributive property holds:

$$\begin{aligned} p \vee (q \wedge r) &= (p \vee q) \wedge (p \vee r); \\ (p \vee q) \wedge r &= (p \wedge r) \vee (q \wedge r), \end{aligned} \quad (4.12)$$

which defines the Boolean character of Q .

5. Lattice of Propositions, Superposition Principle

We get the following

Lemma (5.a). $p \mapsto \mathcal{P}_1(p)$ is a mapping of Q into \mathcal{M} .

In fact, let β be any state belonging to $\overline{\mathcal{P}_1(p)}$: since $\mathcal{P}_1(p) = \mathcal{P}_0(p^\perp)$ we get, by closure definition, $m_\beta(q) = 0$ for every q such that $[\mathcal{P}_0(p^\perp)](q) \cong 0$; of course $[\mathcal{P}_0(p^\perp)](p^\perp) \cong 0$, hence $m_\beta(p^\perp) = 0$, i.e. $\beta \in \mathcal{P}_0(p^\perp) = \mathcal{P}_1(p)$. Thus we find $\mathcal{P}_1(p) \supseteq \overline{\mathcal{P}_1(p)}$: looking at (3.3) it is then proved that $\mathcal{P}_1(p)$ is closed, i.e., $\mathcal{P}_1(p) \in \mathcal{M}$.

The mapping $p \mapsto \mathcal{P}_1(p)$ becomes a one to one correspondence between Q and \mathcal{M} if one adopts the next

Axiom 4.

For every $\mathcal{P} \in \mathcal{M}$ there exists a proposition p such that $\mathcal{P} = \mathcal{P}_1(p)$.

The unicity of this proposition is ensured by Axiom 2. We shall denote by μ the one-to-one mapping which brings \mathcal{M} onto Q ; thus we write $\mathcal{P} \mapsto \mu(\mathcal{P})$, $p \mapsto \mu^{-1}(p) = \mathcal{P}_1(p)$.

Axiom 4 looks trivial in standard classical mechanics: in fact Q , as well as $\mathcal{M} = M$, can be viewed as the set of all subsets of the phase space \mathcal{I} , and, as already noticed, \mathcal{V}_p just represents $\mathcal{P}_1(p)$.

Now we state the relevant

Theorem (5.b). If Q is equipped with the strong ordering, μ is an isomorphism between the lattice \mathcal{M} and Q .

In fact, μ clearly preserves the ordering relation (4.4). Moreover, given any subset \mathcal{Q} of Q , consider the image $\mu^{-1}(\mathcal{Q}) = \{\mathcal{P} \in \mathcal{M} : \mu(\mathcal{P}) \in \mathcal{Q}\}$: the least upper bound $\bigvee_{\mathcal{P} \in \mu^{-1}(\mathcal{Q})} \mathcal{P}$ and the greatest lower bound $\bigwedge_{\mathcal{P} \in \mu^{-1}(\mathcal{Q})} \mathcal{P}$ certainly exist since \mathcal{M} is a complete lattice.

The propositions $\mu(\bigvee_{\mathcal{P} \in \mu^{-1}(\mathcal{Q})} \mathcal{P})$ and $\mu(\bigwedge_{\mathcal{P} \in \mu^{-1}(\mathcal{Q})} \mathcal{P})$ are the least upper bound and the greatest lower bound of the propositions belonging to \mathcal{Q} since μ preserves the ordering relation. This completes the proof.

Hereinafter the ordering in Q will be definitely assumed to be the strong one, so that the previous theorem gives to Q the structure of a complete lattice. Accordingly the set of proposition will be denoted by L , rather than by Q ; the ordering relation will be simply denoted by \leq . The isomorphism between L and \mathcal{M} , is identified⁶ with the “superposition principle” of quantum mechanics.

To equip Q with the strong ordering seems to be a crucial requirement. In fact it has been shown¹⁸ that there are examples of axiomatic structures which meet almost all standard axioms of propositional calculus of quantum mechanics but do not admit the strong ordering and actually have little to do with a description of physical systems.

As already noticed, for classical mechanics \mathcal{M} becomes equal to the Boolean σ -algebra¹⁹ M and the analog of Theorem (5.b) is trivial: the very reason of this is that in classical mechanics we do not admit superpositions of states other than mixtures.

The orthocomplementation $p \mapsto p^\perp$ defined in L determines through the isomorphism μ an orthocomplementation

$$\mu^{-1}(p) = \mathcal{P}_1(p) \mapsto \mu^{-1}(p^\perp) = \mathcal{P}_0(p).$$

From the relation $p \vee p^\perp = 1_L$ we deduce, recalling the definition of least upper bound in \mathcal{M} ,

$$\overline{\mathcal{P}_1(p) \cup \mathcal{P}_0(p)} = P;$$

i.e., the closure of the set $\{\alpha \in P : m_\alpha(p) = 0, 1\}$ covers P , for any $p \in L$.

We also remark that Axiom 4 ensures that the equation $\mathcal{P}(q) \cong 0$, $q \neq 0_L$, occurring in the definition of superposition admits at least a solution q for every non empty $\mathcal{P} \subset P$. In fact, if \mathcal{P} is closed the propositions of the set $\{q \in L : q \leq (\mu(\mathcal{P}))^\perp\}$ exhaust the solutions; if \mathcal{P} is not closed, then $\mathcal{P} \subseteq \overline{\mathcal{P}}$, and the propositions of the set $\{q \in L : q \leq (\mu(\overline{\mathcal{P}}))^\perp\}$ are certainly solutions.

Let $\{p_i\}$ be any set of propositions, i running over the index set I : if there exists a proposition q such that $m_\alpha(q) = \sum_{i \in I} m_\alpha(p_i)$ for all $\alpha \in P$ (hence for all $\alpha \in S$) we shall say that q is the sum of the p_i 's and write $q = \sum_{i \in I} p_i$. Of course the sum q exists only if $\alpha \in \mathcal{P}_1(p_i)$ implies $\alpha \in \mathcal{P}_0(p_j)$ for $j \neq i$, i.e., if $p_i \leq p_j^\perp$ for $i \neq j$. Two propositions p_i, p_j satisfying the symmetric relation $p_i \leq p_j^\perp$ are said to be orthogonal and we write $p_i \perp p_j$. If the sum of mutually orthogonal propositions exists^{19a}, then

$$\sum_{i \in I} p_i = \bigvee_{i \in I} p_i, \quad (p_i \perp p_j, i \neq j). \quad (5.1)$$

In fact, after remarking that $\mathcal{P}_1(\sum_{i \in I} p_i) \supseteq \mathcal{P}_1(p_j)$, $j \in I$, one gets (5.1) if any proposition r such that $r \geq p_j$, $j \in I$, fulfils $r \geq \sum_{i \in I} p_i$. Actually consider the proposition $r^\perp + \sum_{i \in I} p_i$ and write the obvious property $m_\alpha(r^\perp + \sum_{i \in I} p_i) \leq 1$, i.e.,

$$m_\alpha(\sum_{i \in I} p_i) \leq 1 - m_\alpha(r^\perp) = m_\alpha(r),$$

for any state α : hence $\mathcal{P}_1(\sum_{i \in I} p_i) \subseteq \mathcal{P}_1(r)$.

6. Atomicity, State Support

An element b of a partially ordered set E — in particular a lattice — is an atom if (i) $b \neq 0_E$, (ii) $0_E \leq x \leq b$ implies $x = 0_E$ or $x = b$. The set E is said to be atomic if every $x \in E$ admits, at least, one atom b such that $b \leq x$.

The axioms adopted insofar do not imply the atomicity of L , nor the existence of atoms. Birkhoff and Von Neumann¹ initially assumed that for

quantum mechanical lattices the modularity relation

$$q \wedge (r \vee p) = (q \wedge r) \vee p \text{ if } p \leq q, (p, q, r \in L) \quad (6.1)$$

were the substitute of the distributivity (4.12) of Boolean lattices; such an assumption has been recognized to prevent L from being atomic²⁰. On the other hand it is since long known⁴ that an atomicity assumption for L is needed to prove the isomorphism between L and the lattice of closed subspaces of an Hilbert space, thus recovering the standard formulation of quantum mechanics. Actually the modularity condition has to be weakened, as we shall discuss in next Sections.

In classical mechanics the relevance of the atomicity of L is displayed by the following statement^{6, 21, 22}: if L is a separable Boolean σ -algebra (i.e., if the number of freedom degrees of the system is finite) the set P of pure states is non empty if and only if L is atomic: denoting by A the set of all atoms of L and associating to every $a \in A$ the state $\alpha(a)$ through

$$m_{\alpha(a)}(p) = 1 \text{ if } p \geq a, \quad m_{\alpha(a)}(p) = 0 \text{ if } p \not\geq a,$$

it is $P = \{\alpha(a) : a \in A\}$ and there is a one-to-one correspondence between P and A . This statement — which is easily expected from the picture of P and L in the phase space — may be used to show that $\mathcal{M} = M$: in fact, given $\mathcal{P} \subset P$ and a pure state $\beta \notin \mathcal{P}$, let b the atom of L associated to β ; hence $m_{\alpha}(b) = 0, \forall \alpha \in \mathcal{P}$ and $m_{\beta}(b) = 1$, so that β cannot be a superposition of the states of \mathcal{P} .

According to a Gleason's theorem²³ the isomorphism between pure states and atoms of L holds also within the Von Neumann model of quantum mechanics in an Hilbert space. On account of this let us adopt the following atomicity axiom:

Axiom 5.

The subsets of P formed by just one pure state are closed (i.e., belong to \mathcal{M}).

Thus \mathcal{M} is made atomic and the isomorphism μ makes L atomic too. Let us remark that Axiom 5 is something more than just to assume the atomicity of L : in fact, the last would imply, through μ^{-1} , the atomicity of \mathcal{M} , but this does not ensure that the atoms of \mathcal{M} are single pure states.

The isomorphism μ between \mathcal{M} and L determines, by restriction, a one-to-one correspondence, between P and the set A of the atoms of L , which is denoted by π : hence, if $\alpha \in P$, $\pi(\alpha) \in A$.

Given $p \in L$, let \mathcal{A}_p the set of all atoms smaller than p :

$$\mathcal{A}_p = \{a \in A : a \leq p\}.$$

The correspondence π^{-1} associates to \mathcal{A}_p the set of pure states

$$\mathcal{P}_p = \{\pi^{-1}(a) : a \in \mathcal{A}_p\},$$

which, on account of the ordering relation operating in \mathcal{M} (set inclusion), has to coincide with $\mu^{-1}(p) \in \mathcal{M}$. Moreover, Axiom 5 ensures that the elements of \mathcal{P}_p are elements of \mathcal{M} and the rule giving the least upper bound in \mathcal{M} allows to write

$$\mu^{-1}(p) = \bigvee_{a \in \mathcal{A}_p} \pi^{-1}(a);$$

therefore

$$p = \bigvee_{a \in \mathcal{A}_p} a. \quad (6.2)$$

This shows that L (and \mathcal{M}) is not only atomic but even "atomistic", i.e., any proposition p is the least upper bound of the atoms contained in p . Of course it may happen that there exists a subset of \mathcal{A}_p whose least upper bound is p . At this purpose we mention²⁴ that it is always possible to pick out from \mathcal{A}_p a countable sequence $\{a_i; i \in I\}$ of mutually orthogonal atoms such that $p = \bigvee_{i \in I} a_i$. The

countability requirement finds his analog in classical mechanics in the separability of the Boolean σ -algebra; in standard quantum mechanics it corresponds to the separability of the Hilbert space.

For every state α , define the subset of L

$$\mathcal{L}_1(\alpha) = \{p \in L : m_{\alpha}(p) = 1\}; \quad (6.3)$$

comparing Axioms 4 and 5 and recalling (4.10), one sees that $\mathcal{L}_1(\alpha)$ cannot be empty. If there exists a proposition $\sigma(\alpha)$ such that

$$\mathcal{L}_1(\alpha) = \{p \in L : p \geq \sigma(\alpha)\}, \quad (6.4)$$

we shall refer to it as the support of α ²⁵. Clearly, if a support of the state α exists, it is unique. We prove the

Theorem (6,a). The unique support of $\alpha \in P$ is the atom $\pi(\alpha)$; $a \in A$ is the support of the pure state $\pi^{-1}(a)$ only.

In fact $\{\alpha\} = \mathcal{P}_1(\pi(\alpha))$, i.e., $m_{\alpha}(\pi(\alpha)) = 1$, hence any $p \in \mathcal{L}_1(\alpha)$ satisfies $\mathcal{P}_1(p) \supseteq \mathcal{P}_1(\pi(\alpha))$, i.e., $p \geq \pi(\alpha)$: this proves that $\pi(\alpha) = \sigma(\alpha)$. Since the correspondence π is one-to-one, clearly $a \in A$ is the support of $\pi^{-1}(a)$; moreover the atom a cannot be the support of states other than $\pi^{-1}(a)$ since we know that $\mathcal{P}_1(a)$ has $\pi^{-1}(a)$ as unique element.

For completeness we now prove the next

Theorem (6,b).

(i) Given the mixture $\alpha = \sum_{i \in I} t_i \alpha_i$, $\alpha_i \in P$ we have

$$\sigma(\alpha) = \bigvee_{i \in I} \pi(\alpha_i);$$

(ii) given the non zero proposition $p = \bigvee_{i \in I} a_i$,

where the atoms a_i 's are such that $a_i \perp a_j$ for $i \neq j$, if the sum $\sum_{i \in I} a_i$ exists, then p is the support of all the mixtures $\sum_{j \in I} t_j \pi^{-1}(a_j)$.

Proof:

(i) From (4.10) and Theorem (6, a) it follows that if q is such that $m_\alpha(q) = 1$ then $q \geq \pi(\alpha_i)$, $\forall i \in I$: hence $\bigvee_{i \in I} \pi(\alpha_i)$ is the support of α ;

(ii) Let $\alpha = \sum_{j \in I} t_j \pi^{-1}(a_j)$. If $q \geq p$ we have, from (5.1) and Theorem (6,a),

$$\begin{aligned} m_\alpha(\bigvee_{i \in I} a_i) &= \sum_{j \in I} t_j m_{\pi^{-1}(a_j)}(\bigvee_{i \in I} a_i) \\ &= \sum_{i, j \in I} t_j m_{\pi^{-1}(a_j)}(a_i) = \sum_{i, j \in I} t_j \delta_{ij} = 1, \quad \text{hence} \\ m_\alpha(q) &= 1; \text{ conversely, if } m_\alpha(q) = 1 \text{ we have, from} \\ (4.10) \text{ and Theorem (6,a), } q &\geq a_i, \quad \forall i \in I, \text{ hence} \\ q &\geq p. \text{ This completes the proof.} \end{aligned}$$

The following theorem shows the equivalence between the Axiom 5 and an alternative atomicity assumption proposed by Gudder^{17, 26}:

Theorem (6,c). The sets $\{\alpha\}$, which have $\alpha \in P$ as unique element, are closed if and only if $\mathcal{L}_1(\alpha) \subseteq \mathcal{L}_1(\beta)$ implies $\alpha = \beta$.

In fact, let $\beta \in \{\alpha\}$ and choose any $p \in \mathcal{L}_1(\alpha)$; since $m_\alpha(p^\perp) = 0$ we get, by closure definition, $m_\beta(p^\perp) = 0$, i.e., $p \in \mathcal{L}_1(\beta)$: hence $\mathcal{L}_1(\alpha) \subseteq \mathcal{L}_1(\beta)$. If $\mathcal{L}_1(\alpha) \subseteq \mathcal{L}_1(\beta)$ implies $\alpha = \beta$ we get $\{\alpha\} = \{\beta\}$. Conversely, suppose $\{\alpha\}, \{\beta\} \in \mathcal{M}$ for any given $\alpha, \beta \in P$; from Theorem (6,a) we know that the characterization (6.4) may be used, hence we have the implications

$$\begin{aligned} \mathcal{L}_1(\alpha) \subseteq \mathcal{L}_1(\beta) &\Rightarrow \{p \in L : p \geq \pi(\alpha)\} \subseteq \\ &\subseteq \{p \in L : p \geq \pi(\beta)\} \Rightarrow \pi(\alpha) \geq \pi(\beta), \end{aligned}$$

then $\pi(\alpha) = \pi(\beta)$ since $\pi(\alpha)$ and $\pi(\beta)$ are atoms. This completes the proof.

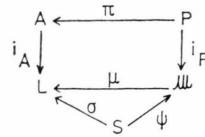
The last theorem ensures that there are no two pure states such that $\mathcal{L}_1(\alpha) \subset \mathcal{L}_1(\beta)$: let us remark that if such an inclusion relation would exist, the set $\{\mathcal{L}_1(\alpha) : \alpha \in P\}$ would become partially ordered and one would induce an unphysical partial ordering relation among states. In other words, Axiom 5 is

equivalent to state that $\mathcal{L}_1(\alpha)$ is a prime ideal of L , for any $\alpha \in P$. On account of Theorem (6,a), the support of a pure state α is the greatest lower bound of the propositions belonging to $\mathcal{L}_1(\alpha)$:

$$\sigma(\alpha) = \pi(\alpha) = \bigvee_{p \in \mathcal{L}_1(\alpha)} p, \quad \alpha \in P. \quad (6.5)$$

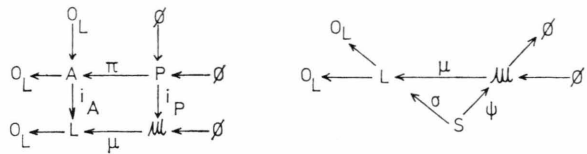
The one-to-one correspondence between pure states and the prime ideals $\mathcal{L}_1(\alpha)$, or the associated atoms obtained by (6.5), has been proposed as definition of pure state by Jauch and Piron²⁷, in a scheme which does not make use of the isomorphism between \mathcal{M} and L .

Let us summarize the various correspondences defined insofar, by the following diagram:



where i_A and i_P are the canonical inclusions of A in L and, respectively, of P in \mathcal{M} (Axiom 5), while ψ may be defined as follows: if $\alpha = \sum_{i \in I} t_i \alpha_i$, $\alpha_i \in P$,

then $\psi(\alpha) = \overline{\{\alpha_i\}}$. From this diagram we can extract, with standard algebraic notations, the two exact and commutative diagrams.



7. Operations, Compatible Propositions

In the previous Sections the notion of proposition has been used only to study the properties of the probability measure $m_\alpha(p)$; to complete the physical meaning of propositions we must now study how the state α is transformed by the interaction of the system with the experimental apparatus which is used to perform the proposition p . Notice that our definition of equal propositions [see (2.3) and Axiom 2] leaves open the possibility of performing the proposition p by different experimental procedures which transform the state of the system in different ways.

Given $p \in L$, $\alpha \in S$, denote by $\Omega_{p,j}\alpha$, $j = 1, 2, \dots$, the final state resulting from the "yes" output of

j 'th prozedure used to measure p^{28} ; the domain of the map $\Omega_{p,j}$ will be denoted by $D[\Omega_{p,j}]$ and its range by $R[\Omega_{p,j}]$. Two maps $\Omega_{p,j}$, $\Omega_{q,k}$ are equal if $D[\Omega_{p,j}] = D[\Omega_{q,k}]$

and $\Omega_{p,j}\alpha = \Omega_{q,k}\alpha, \forall \alpha \in D[\Omega_{p,j}]$.

The composition $\Omega_{p,j} \cdot \Omega_{q,k}$ of $\Omega_{p,j}$ with $\Omega_{q,k}$ is the map defined by

$$\begin{cases} D[\Omega_{p,j} \cdot \Omega_{q,k}] = \{\alpha \in D[\Omega_{q,k}] : \Omega_{q,k}\alpha \in D[\Omega_{p,j}]\}, \\ (\Omega_{p,j} \cdot \Omega_{q,k})\alpha = \Omega_{p,j}(\Omega_{q,k}\alpha), \forall \alpha \in D[\Omega_{p,j} \cdot \Omega_{q,k}]. \end{cases} \quad (7.1)$$

According to the general background of quantum theory of measurement we first assume that, for some j 's, $\Omega_{p,j}$ is pure, i.e. transforms pure states into pure states. Such maps correspond to experimental apparatus which do not increase the entropy of the incoming state: hence the transformation law of mixture states is entirely determined by the transformation law of pure states. Thus the domain (and the range) of pure maps will be restricted to subsets of P . We next assume that, for some j 's, $\Omega_{p,j}$ is pure and of first kind, i.e.

- (i) $D[\Omega_{p,j}] = C\mu^{-1}(p^\perp)$,
- (ii) $m_{\Omega_{p,j}\alpha}(p) = 1, \forall \alpha \in D[\Omega_{p,j}]$,
- (iii) $m_\alpha(p) = 1 \Leftrightarrow \Omega_{p,j}\alpha = \alpha$,

where $C\mu^{-1}(p^\perp)$ stands for the complement of $\mu^{-1}(p^\perp)$, (notice that the right to left implication in (iii) is a consequence of the other statements). The first requirement (7.2) states that $\Omega_{p,j}$ is not defined on the states such that $m_\alpha(p) = 0$: for these states the measurement of p is expected to annihilate the state (think, e.g., to a light polarizer); the last two requirements clearly refer to the usual distinction between first kind and second kind measurements. One cannot expect that the pure, first kind map associated to p is unique: the measurement of a degenerate eigenvalue could provide a counterexample. We then adopt the following

Axiom 6.

Among the pure, first kind maps associated to any proposition p , there exists a unique map, to be simply denoted by Ω_p , such that $m_\alpha(p) = 1$ implies $m_{\Omega_q\alpha}$ and that all of shem $(p) = 1$ whenever $\Omega_p \cdot \Omega_q = \Omega_q \cdot \Omega_p$.

According to standard terminology we can say that it is possible to measure any proposition by pure, first kind, ideal measurements* transform the state of the system under the same map. Herein-

after, by measurement of p we shall always mean the pure, first kind, ideal measurement of p .

Two propositions p, q satisfying the property $\Omega_p \cdot \Omega_q = \Omega_q \cdot \Omega_p$ are said to be compatible; Axiom 6 ensures that if the output of p is certainly yes for the state α , it remains so after the measurement of any proposition q compatible with p .

From the requirement (ii) of (7.2) it follows $R[\Omega_p] \subseteq \mathcal{P}_1(p)$, while (iii) gives $R[\Omega_p] \supseteq \mathcal{P}_1(p)$ so that

$$R[\Omega_p] = \mu^{-1}(p). \quad (7.3)$$

Let us remark (see (7.2) and ignore hereinafter the index j) that $D[\Omega_{0_L}] = \emptyset$ so that Ω_{0_L} is the zero map; moreover $D[\Omega_{1_L}] = P = R[\Omega_{1_L}]$ and $\Omega_{1_L}\alpha = \alpha, \forall \alpha \in P$, so that Ω_{1_L} is the identical map. From the composition law (7.1) we obtain

$$\begin{aligned} \Omega_{0_L} \cdot \Omega_p &= \Omega_p \cdot \Omega_{0_L} = \Omega_{0_L}; \\ \Omega_{1_L} \cdot \Omega_p &= \Omega_p \cdot \Omega_{1_L} = \Omega_p, \quad \forall p \in L. \end{aligned} \quad (7.4)$$

Consider the composition $\Omega_{p_1} \cdot \Omega_{p_2} \cdot \dots \cdot \Omega_{p_n}$ of a finite number of maps: it represents the measurement of the proposition p_n , followed by the measurement of p_{n-1} and so on, lasting with the measurement of p_1 . Such compositions of maps will be shortly called operations. It may happen that two different sets of propositions, say (p_1, \dots, p_n) , (q_1, \dots, q_m) , give rise to the same operation: then the following axiom will be adopted²⁹:

Axiom 7.

$$\begin{aligned} \Omega_{p_1} \cdot \dots \cdot \Omega_{p_n} &= \Omega_{q_1} \cdot \dots \cdot \Omega_{q_m} \Rightarrow \\ \Omega_{p_n} \cdot \dots \cdot \Omega_{p_1} &= \Omega_{q_m} \cdot \dots \cdot \Omega_{q_1}. \end{aligned}$$

Let

$$T_\Omega = \{\Omega_{p_1} \cdot \dots \cdot \Omega_{p_n}; n \text{ finite}\} \quad (7.5)$$

be the set of all operations. The composition of maps gives to T_Ω the structure of a semigroup; according to (7.4), the maps Ω_{0_L} and Ω_{1_L} are the zero and unit elements of the semigroup. On account of Axiom 7 the mapping

$$\Omega_{p_1} \cdot \dots \cdot \Omega_{p_n} \mapsto (\Omega_{p_1} \cdot \dots \cdot \Omega_{p_n})^\dagger = \Omega_{p_n} \cdot \dots \cdot \Omega_{p_1} \quad (7.6)$$

is well defined and is an involutive automorphism of T_Ω : in fact, for any $x, y \in T_\Omega$ we have

$$(x^\dagger)^\dagger = x, \quad (x \cdot y)^\dagger = y^\dagger \cdot x^\dagger. \quad (7.7)$$

Given two ordered propositions, say $p \leq q$, we get, from (7.3) and (7.2),

$$R[\Omega_p] \subseteq R[\Omega_q] \subseteq D[\Omega_q],$$

hence $D[\Omega_q \cdot \Omega_p] = D[\Omega_p]$,

then $\Omega_q \cdot \Omega_p\alpha = \Omega_p\alpha, \forall \alpha \in D[\Omega_p]$,

so that we conclude $p \leq q \Rightarrow \Omega_q \cdot \Omega_p = \Omega_p$. By use of Axiom 7, $p \leq q \Rightarrow \Omega_p \cdot \Omega_q = \Omega_p$. Conversely, if $\Omega_p \cdot \Omega_q = \Omega_p$, we have

$$D[\Omega_p] = D[\Omega_p \cdot \Omega_q] \subseteq D[\Omega_q]$$

hence, by (7.2), $p^\perp \geq q^\perp$, i.e., $p \leq q$. Thus it is proved that²⁹

$$p \leq q \Leftrightarrow \Omega_p \cdot \Omega_q = \Omega_p \Leftrightarrow \Omega_q \cdot \Omega_p = \Omega_p. \quad (7.8)$$

As a particular case

$$\Omega_p \cdot \Omega_p = \Omega_p, \quad \forall p \in L. \quad (7.9)$$

From (7.8) one also deduces

$$p \leq q \Rightarrow \Omega_p \cdot \Omega_q = \Omega_q \cdot \Omega_p, \quad (7.10)$$

i.e., if $p \leq q$ then p is compatible with q : this statement is nothing else the so called “weak modularity” which is usually given, in a different language, as an independent axiom. This point will be discussed further in the sequel.

Let us recall that in a semigroup T equipped with an involution $x \mapsto x^\dagger$, the elements $f \in T$ which fulfil the equalities $f \cdot f = f = f^\dagger$ are called projections. The set $N(T)$ of the projections of T is partially ordered by putting $f \leq e$ whenever $f \cdot e = f$, $e, f \in N(T)$: in fact this relation is reflexive ($f \cdot f = f$), transitive ($g \cdot f = g, f \cdot e = f \Rightarrow g \cdot e = g \cdot f \cdot e = g$) and antisymmetric ($f \cdot e = f, e \cdot f = e \Rightarrow e = (e \cdot f)^\dagger = (f^\dagger \cdot e^\dagger)^\dagger = (f \cdot e)^\dagger = f$). If T has a zero element 0_T and if a mapping $x \mapsto x'$ of T into $N(T)$ is defined such that

$$\{y \in T : x \cdot y = 0_T\} = \{y \in T : x' \cdot y = y\}, \quad (7.11)$$

then T is a Baer semigroup. A projection $f \in N(T)$ is closed under the mapping $x \mapsto x'$ if $(f')' = f$; let

$$\mathcal{N}(T) = \{f \in N(T) : (f')' = f\} \quad (7.12)$$

be the set of closed projections.

Consider now to the semigroup T_Ω equipped with the involution (7.6). Looking at (7.8) one sees that $p \mapsto \Omega_p$ is a mapping of L into $N(T_\Omega)$ which preserves the ordering relation. To make T_Ω a Baer semigroup it is sufficient to adopt the following

Axiom 8.

$$CD[\Omega_{p_1} \dots \Omega_{p_n}] \in \mathcal{M}, \quad \forall p_1, \dots, p_n \in L.$$

This axiom extends to any n a property met for $n = 1$: in fact, from (7.2), $CD[\Omega_p] = \mu^{-1}(p^\perp)$. Therefore, recalling the isomorphism between L and \mathcal{M} , each $x \in T_\Omega$ determines a unique proposition

q_x such that

$$D[x] = C\mu^{-1}(q_x^\perp). \quad (7.13)$$

Now we define the mapping

$$x \mapsto x' = \Omega_{q_x}, \quad x \in T_\Omega, \quad (7.14)$$

and show that it fulfills the requirement (7.11)²⁹. In fact $x \cdot y = \Omega_{0_L}$ means $y \alpha \in CD[x]$, $\forall \alpha \in D[y]$, i.e., $y \alpha \in \mu^{-1}(q_x^\perp)$, $\forall \alpha \in D[y]$; by use of the last statement (7.2) we then have that $x \cdot y = \Omega_{0_L}$ is equivalent to $x' \cdot y \alpha = y \alpha$, $\forall \alpha \in D[y]$ which implies $D[x' \cdot y] = D[y]$; hence $x \cdot y = \Omega_{0_L} \Leftrightarrow x' \cdot y = y$. Thus T_Ω is a Baer semigroup.

The mapping $p \mapsto \Omega_p$ is a one-to-one correspondence of L onto $\mathcal{N}(T_\Omega)$: in fact comparing (7.2), (7.13) and (7.14) it is seen that $\Omega'_p = \Omega_{p^\perp}$, hence $\Omega_p \in \mathcal{N}(T_\Omega)$; conversely, we use a general property of Baer semigroups stating that f belongs to $\mathcal{N}(T)$ if and only if $f = x'$ for some $x \in T$ ²⁴, so that the relation (7.14) and the unicity of q_x complete the argument. Furthermore, $p \mapsto \Omega_p$ is an isomorphism between the complete lattice L and $\mathcal{N}(T_\Omega)$: in fact the right hand side of (7.8) is precisely just the ordering relation defined in $\mathcal{N}(T_\Omega)$; hence $\mathcal{N}(T_\Omega)$ is a complete lattice. The orthocomplementation $p \mapsto p^\perp$ in L determines, through the isomorphism, the orthocomplementation $\Omega_p \mapsto (\Omega_p)^\perp = \Omega_{p^\perp}$ in $\mathcal{N}(T_\Omega)$ which coincides with the restriction to $\mathcal{N}(T_\Omega)$ of the mapping (7.14). The explicit form of the greatest lower bound in $\mathcal{N}(T_\Omega)$ is given by²⁴

$$\Omega_p \wedge \Omega_q = \Omega_p \cdot (\Omega_{q^\perp} \cdot \Omega_p)^\perp = \Omega_{p \wedge q}$$

while $\Omega_p \vee \Omega_q = (\Omega_{p^\perp} \wedge \Omega_{q^\perp})^\perp = \Omega_{p \vee q}$,

Owing to the isomorphism between L and $\mathcal{N}(T_\Omega)$ all properties of closed projections of Baer semigroups may be transferred to L ²⁹. The following is relevant²⁴: $\mathcal{N}(T_\Omega)$ is an “orthomodular” lattice and

$$\begin{aligned} \Omega_p \cdot \Omega_q &= \Omega_q \cdot \Omega_p \Leftrightarrow \\ &\begin{cases} \Omega_p = (\Omega_p \wedge \Omega_q) \vee (\Omega_p \wedge \Omega_{q^\perp}), \\ \Omega_q = (\Omega_q \wedge \Omega_p) \vee (\Omega_q \wedge \Omega_{p^\perp}); \end{cases} \end{aligned} \quad (7.15)$$

moreover $\Omega_p \cdot \Omega_q = \Omega_q \cdot \Omega_p$ implies $\Omega_p \wedge \Omega_q = \Omega_p \cdot \Omega_q$. Denoting by $p \sim q$ the compatibility relation $\Omega_p \cdot \Omega_q = \Omega_q \cdot \Omega_p$, the relation (7.15) reads

$$p \sim q \Leftrightarrow \begin{cases} p = (p \wedge q) \vee (p \wedge q^\perp), \\ q = (q \wedge p) \vee (q \wedge p^\perp). \end{cases} \quad (7.16)$$

According to (7.10), $p \leq q$ implies $p \sim q$; hence, remarking that $p \leq q$ gives $p \wedge q = p$ and

$p \wedge q^\perp = 0_L$, we get

$$p \leq q \Rightarrow q = p \vee (q \wedge p^\perp). \quad (7.17)$$

With reference to the isomorphism μ between \mathcal{M} and L we may visualize the various possibilities arising from (7.16) and (7.17). Reminding that the greatest lower bound in \mathcal{M} is just the set intersection we have that $p, q \in L$ are

non compatible if:

- (i) $\mu^{-1}(p) \cap \mu^{-1}(q) = \emptyset$, $\mu^{-1}(p) \cap \mu^{-1}(q^\perp) \neq \mu^{-1}(p)$,
- (ii) $\mu^{-1}(p) \cap \mu^{-1}(q) \neq \mu^{-1}(p)$, $\mu^{-1}(p) \cap \mu^{-1}(q^\perp) = \emptyset$;

compatible if:

- (i) $\mu^{-1}(p) \cap \mu^{-1}(q) = \emptyset$, $\mu^{-1}(p) \cap \mu^{-1}(q^\perp) = \mu^{-1}(p)$,
- (ii) $\mu^{-1}(p) \cap \mu^{-1}(q) = \mu^{-1}(p)$, $\mu^{-1}(p) \cap \mu^{-1}(q^\perp) = \emptyset$;

while nothing is said, a priori, if

$$\mu^{-1}(p) \cap \mu^{-1}(q) \neq \emptyset \neq \mu^{-1}(p) \cap \mu^{-1}(q^\perp).$$

Let us remark that (7.17) implies that if p is strictly less than q , then, for some $\alpha \in P$, $m_\alpha(p) = 0$, $m_\alpha(q) = 1$. The relation (7.17) coincides with the so called property of weak modularity which is usually presented as an independent axiom: the name is motivated by the fact that it derives as a particular case from the modularity relation (6.1) letting $r = p^\perp$. As it is implicit in our derivation, weak modularity does not conflict with atomicity, contrary to what happen for modularity.

We believe that the definition of compatibility here used is more physical than the one usually found in the literature.

We also remark that from (7.16) one obviously get

$$p \sim q \Leftrightarrow p \sim q^\perp \Leftrightarrow p^\perp \sim q \Leftrightarrow p^\perp \sim q^\perp. \quad (7.18)$$

Moreover^{24, 6} p is compatible with q if and only if the sublattice generated by $\{p, p^\perp, q, q^\perp\}$ is distributive³⁰: thus

$$r_1 \wedge (r_2 \vee r_3) = (r_1 \wedge r_2) \vee (r_1 \wedge r_3) \quad (7.19)$$

if $r_1, r_2, r_3 \in \{p, p^\perp, q, q^\perp\}$ with $p \sim q$.

The compatibility conditions simplify considerably if atoms are concerned. In fact, given $a \in A$, $q \in L$, we have $a \sim q$ if and only if either $\pi^{-1}(a) \subseteq \mu^{-1}(q)$ or $\pi^{-1}(a) \subseteq \mu^{-1}(q^\perp)$; if $a, b \in A$ we have $a \sim b$ if and only if $\pi^{-1}(a) \subseteq \mu^{-1}(b^\perp)$.

Let us finally remark that in the classical case $\mathcal{M} = M$, $\mu^{-1}(q^\perp) = C \mu^{-1}(q)$, $\mu^{-1}(p) \vee \mu^{-1}(q) = \mu^{-1}(p) \cup \mu^{-1}(q)$ so that any two propositions are compatible.

8. The Covering Law

Let α be any pure state and consider the support $\sigma(\alpha) = \pi(\alpha)$, (see Theorem (6,a)). For every $p \in L$ we have $\sigma(\alpha)^\perp \wedge p \leq p$, hence, by (7.10), $\sigma(\alpha)^\perp \wedge p \sim p$, hence, by (7.18) and (4.7), $\sigma(\alpha) \vee p^\perp \sim p$. Being $\sigma(\alpha) \vee p^\perp \geq \sigma(\alpha)$, we have, by support definition, $m_\alpha(\sigma(\alpha) \vee p^\perp) = 1$, $\forall \alpha \in P$: restricting α to $D[\Omega_p]$ and applying Axiom 6 we get

$$m_{\Omega_p \alpha}(\sigma(\alpha) \vee p^\perp) = 1, \quad \forall \alpha \in D[\Omega_p];$$

hence $\sigma(\Omega_p \alpha) \leq \sigma(\alpha) \vee p^\perp$, $\forall \alpha \in D[\Omega_p]$.

Moreover from (ii) of (7.2) we get $\sigma(\Omega_p \alpha) \leq p$, $\forall \alpha \in D[\Omega_p]$; thus

$$\sigma(\Omega_p \alpha) \leq (\sigma(\alpha) \vee p^\perp) \wedge p, \quad \alpha \in D[\Omega_p], \quad p \in L. \quad (8.1)$$

Actually this result is completed by the following

Theorem (8,a). For every $p \in L$ and every $\alpha \in D[\Omega_p]$, we have $\sigma(\Omega_p \alpha) = (\sigma(\alpha) \vee p^\perp) \wedge p$. (Notice that when α spans $D[\Omega_p]$, $\sigma(\alpha)$ spans the set of atoms satisfying $a \not\leq p^\perp$.)

Proof. Suppose that, for some proposition \bar{p} and some state $\bar{\alpha}$ one has strictly

$$\sigma(\Omega_{\bar{p}} \bar{\alpha}) < (\sigma(\bar{\alpha}) \vee \bar{p}^\perp) \wedge \bar{p}, \quad \bar{\alpha} \in D[\Omega_{\bar{p}}];$$

then there exists an atom b such that

$$\sigma(\Omega_{\bar{p}} \bar{\alpha}) \neq b < (\sigma(\bar{\alpha}) \vee \bar{p}^\perp) \wedge \bar{p}. \quad (8.2)$$

Let us define a map $\Theta_{\bar{p}}$ of P into P by

$$D[\Theta_{\bar{p}}] = D[\Omega_{\bar{p}}],$$

$$\Theta_{\bar{p}} \alpha = \Omega_{\bar{p}} \alpha \quad \text{if} \quad \alpha \neq \bar{\alpha}, \quad \Theta_{\bar{p}} \bar{\alpha} = \pi^{-1}(b).$$

This map has the following properties:

- (i) $\Theta_{\bar{p}}$ is pure by construction;
- (ii) from $\sigma(\Theta_{\bar{p}} \bar{\alpha}) = b < [\sigma(\bar{\alpha}) \vee \bar{p}^\perp] \wedge \bar{p} \leq \bar{p}$ it follows that $m_{\Theta_{\bar{p}} \bar{\alpha}}(\bar{p}) = 1$, so that $\Theta_{\bar{p}}$ fulfills the second requirement (7.2);
- (iii) $\Theta_{\bar{p}}$ fulfills the third requirement (7.2) by construction: in fact it is $m_{\bar{\alpha}}(\bar{p}) < 1$, for if it were $m_{\bar{\alpha}}(\bar{p}) = 1$ then $\sigma(\bar{\alpha}) \leq \bar{p}$, i.e., $\sigma(\bar{\alpha}) \sim \bar{p}$, and recalling (7.18) and (7.19) one would deduce $[\sigma(\bar{\alpha}) \vee \bar{p}^\perp] \wedge \bar{p} = \sigma(\bar{\alpha})$ contrary to the assumption (8.2) which prevents $(\sigma(\bar{\alpha}) \vee \bar{p}^\perp) \wedge \bar{p}$ from being an atom³³;

(iv) if q is compatible with \bar{p} then $m_\alpha(q) = 1$ implies $m_{\Theta_{\bar{p}} \alpha}(q) = 1$: this property is ensured by construction if $\alpha \neq \bar{\alpha}$; if $\alpha = \bar{\alpha}$, the relation $m_{\bar{\alpha}}(q) = 1$ implies, by support definition, $\sigma(\bar{\alpha}) \leq q$, hence we have $\sigma(\Theta_{\bar{p}} \bar{\alpha}) < (\sigma(\bar{\alpha}) \wedge \bar{p}^\perp) \wedge \bar{p} \leq (q \vee \bar{p}^\perp) \wedge \bar{p} =$

$q \wedge \bar{p} \leq q$, where the equality $(q \vee \bar{p}^\perp) \wedge \bar{p} = q \wedge \bar{p}$ is motivated by (7.18) and (7.19): therefore $m_{\Theta_{\bar{p}}\bar{a}}(q) = 1$.

This set of properties met by $\Theta_{\bar{p}}$ shows that $\Theta_{\bar{p}}$, as well as $\Omega_{\bar{p}}$ is a pure, first kind, ideal map associated to \bar{p} . This is a contradiction because Axiom 6 states that such a map is unique. Therefore we have to conclude that the inequality

$$\sigma(\Omega_p \alpha) < [\sigma(\alpha) \vee p^\perp] \wedge p$$

can never occur. The theorem is thus proved.

Let us remark that in the classical case, owing to the distributivity of L , one gets $\sigma(\Omega_p \alpha) = \sigma(\alpha) \wedge p$.

On account of Theorem (6,a), we can also formulate the Theorem (8,a) by saying

$$\pi(\Omega_p \alpha) = (\pi(\alpha) \vee p^\perp) \wedge p, \quad \forall p \in L, \quad \forall \alpha \in D[\Omega_p]. \quad (8.3)$$

This puts in an explicit form the change produced on the state of the system by the measurement of the proposition p .

Since the correspondence π associates atoms to pure states we can also deduce, from Theorem (8,a):

$$(a \vee p^\perp) \wedge p \in A, \quad \forall p \in L, \quad \forall a \in A: a \not\leq p^\perp. \quad (8.4)$$

This property is usually called the ‘‘covering law’’. It has been proved^{24, 34} that (8.4) is fully equivalent to the following property: if $a \in A$ and $p \in L$ there is no proposition q satisfying $p < q < p \vee a$. Taking into account the atomistic structure of L , it turns out that either $p \vee a$ contains only one atom (a itself) more than p or any set containing more than all atoms of p but less than all atoms of $p \vee a$ does not represent a proposition. Looking at the isomorphism between L and \mathcal{M} we can say that either $\mu^{-1}(p \vee a)$ contains only the state $\pi^{-1}(a)$ more than $\mu^{-1}(p)$ does or any set containing more states than $\mu^{-1}(p)$ but less states than $\mu^{-1}(p \vee a)$ is not closed.

The covering law was first introduced⁴ as a technical independent assumption in order to prove the isomorphism between L and the lattice of closed subspaces of an Hilbert space (see Section 10). As we have seen, it may be deduced in the framework of the preceding axioms: the very peculiar role of the unicity requirement involved in Axiom 6 has been stressed by Ochs³⁵.

Inspection of (8.3) suggests associating to the proposition p the map $q \mapsto (q \vee p^\perp) \wedge p$ of L onto L . Pool²⁵ has shown that such maps are the closed projections of the Baer semigroup of residuated appli-

cations³⁶ of L . An application φ of L in L is residuated if (i) $p \leq q \Rightarrow \varphi(p) \leq \varphi(q)$, (ii) the set $\{p \in L: \varphi(p) \leq q\}$ is non-empty and has a maximum element for all $q \in L$. The set Y_L of residuated applications of L is a semigroup with respect to usual composition of applications $(\varphi \cdot \psi)(p) = \varphi(\psi(p))$. It becomes a Baer semigroup (see Section 7) with respect to the involution $\varphi \mapsto \varphi^\dagger$ defined by

$$\varphi^\dagger(q) = [\max\{p \in L: \varphi(p) \leq q^\perp\}]^\perp,$$

and with respect to the mapping $\varphi \mapsto \varphi'$ defined by^{36, 37}

$$\varphi'(q) = [q \vee \varphi^\dagger(1_L)] \wedge [\varphi^\dagger(1_L)]^\perp.$$

Moreover²⁵ the closed projections (see (7.12)) of Y_L form a lattice $\mathcal{N}(Y_L)$ isomorphic to L : the image of $p \in L$ in $\mathcal{N}(Y_L)$ is precisely the map

$$q \mapsto \varphi_p(q) = (q \vee p^\perp) \wedge p.$$

Hence the Theorem (8,a) can also be formulated by saying that $\sigma(\Omega_p \alpha) = \varphi_p(\sigma(\alpha))$.

Comparing reference²⁵ and the Theorem (8,a) one can see that the mapping

$$\Omega_{p_1} \cdot \Omega_{p_2} \cdot \dots \cdot \Omega_{p_n} \mapsto \varphi_{p_1} \cdot \varphi_{p_2} \cdot \dots \cdot \varphi_{p_n}$$

is an homomorphism of the Baer semigroup T_Ω (see (7.5)) into the Baer semigroup Y_L .

9. Irreducible Proposition Systems, Superselection Rules

According to the axiom structure given in the previous Sections, the set of propositions of a quantum system is a complete, orthocomplemented, atomistic and weakly modular lattice for which the covering law holds. For brevity such a lattice will be called a proposition system and still denoted by L . Moreover, propositions systems are isomorphic to the set \mathcal{M} of closed subsets of P . In this Section we shall further characterize the structure of proposition systems, making use of mathematical properties of lattices which may be found in specialized literature²⁴.

If $p, q \in L$ we shall write $p^\nabla q$ if

$$(r \vee p) \wedge q = r \wedge q, \quad \forall r \in L.$$

Given a subset \mathcal{L} of L we shall denote

$$\mathcal{L}^\nabla = \{p \in L: p^\nabla r, \quad \forall r \in \mathcal{L}\};$$

given a sequence $\mathcal{L}_i, i \in I$, of subsets of L , each containing 0_L , we shall say that L is the

direct sum of the \mathcal{L}_i 's and write $L = \bigcup_{i \in I} \mathcal{L}_i$ if (i) every $p \in L$ may be written as $p = \bigvee_{i \in I} p_i$, $p_i \in \mathcal{L}_i$, (ii) $\mathcal{L}_i \subseteq \mathcal{L}_j^\vee$ for $i \neq j$.

The centre $Z(L)$ of L is the set of elements of L which are compatible with every $p \in L$; an element z is said to be central if $z \in Z(L)$. It is easily proved that $Z(L)$ is a Boolean sublattice of L . L is said to be irreducible (or coherent) if $Z(L) = \{0_L, 1_L\}$.

Let a, b be two atoms of L ; we introduce an equivalence relation in the set A of atoms of L putting $a \approx b$ if there exists a third atom $c \neq a, b$ such that $c < a \vee b$; the atoms a and b are then called projective. The equivalence class associated to $a \in A$ will be denoted by $[a]$, i.e., $[a] = \{b \in A : b \approx a\}$. Consider now the set of equivalence classes: denote by I the index set which labels them and let z_i , $i \in I$, be the least upper bound of the atoms belonging to the i -th equivalence class. The following may be proved: (i) $z_i \in Z(L)$, (ii) z_i is an atom with respect to the Boolean lattice $Z(L)$, (iii) $z_i \wedge z_j = 0_L$ if $i \neq j$ while $\bigvee_{i \in I} z_i = 1_L$, (iv) any element of $Z(L)$ may be written as $\bigvee_{i \in \mathcal{J}} z_i$ where \mathcal{J} is a suitable subset of I . Moreover, putting

$$\mathcal{L}[0, z_i] = \{p \in L : p \leq z_i\},$$

it is proved that $\mathcal{L}[0, z_i]$ is irreducible for all $i \in I$ and

$$L = \bigcup_{i \in I} \mathcal{L}[0, z_i].$$

From these results we remark that two atoms $a, b \in A$ belong to the same term $\mathcal{L}[0, z_i]$ if and only if $a \approx b$; furthermore L is irreducible if and only if any two atoms a, b are such that $a \approx b$.

We now come to corresponding properties of pure states. If α is a pure state defined in $\mathcal{L}[0, z_i]$, (i.e., if $m_\alpha(p)$ is defined for $p \in \mathcal{L}[0, z_i]$), one determines a state α^\sim defined in L by setting

$$m_{\alpha^\sim}(p) = m_\alpha(p \wedge z_i), \quad p \in L;$$

then the following is true:

- (i) α^\sim is a pure state defined in L ;
- (ii) the sets

$\mathcal{P}_i = \{\alpha^\sim : \alpha \text{ is a pure state defined in } \mathcal{L}[0, z_i]\}$,
are disjoint subsets of P and $P = \bigcup_{i \in I} \mathcal{P}_i$.

This relevant results is due to Varadarajan⁶; we omit the proof.

Consider two atoms a_i, a_j belonging, respectively, to $\mathcal{L}[0, z_i]$ and $\mathcal{L}[0, z_j]$, $i \neq j$; thus they belong to

different equivalence classes and are not projective. Hence there is no third atom $c \neq a_i, a_j$ contained in $a_i \vee a_j$; with reference to the isomorphism between L and \mathcal{M} we deduce that the set $\{\pi^{-1}(a_i), \pi^{-1}(a_j)\}$ is closed, i.e., there is no pure state which is superposition of $\pi^{-1}(a_i)$ and $\pi^{-1}(a_j)$: all superpositions of the states $\pi^{-1}(a_i), \pi^{-1}(a_j)$ are mixtures. Clearly $\pi^{-1}(a_i) \in \mathcal{P}_i$, $\pi^{-1}(a_j) \in \mathcal{P}_j$; therefore we deduce that if one superposes states belonging to the same \mathcal{P}_i then one certainly gets new pure states, conversely if one superposes states belonging to different \mathcal{P}_i 's then only mixtures are obtained. In a sense, the states belonging to different \mathcal{P}_i 's behave as classical states. In physics such a situation is referred to as the existence of superselection rules. If we consider quantum systems not admitting superselection rules then we must require L to be irreducible.

The isomorphism between L and \mathcal{M} — what we have called the superposition principle — plays the essential role in the previous discussion. In the literature⁵ one also finds the following version of the superposition principle: if a_1, a_2 are any two atoms of L then there exists a third atom a_3 such that $a_1 \vee a_2 = a_1 \vee a_3 = a_2 \vee a_3$. If this statement is adopted, then of course any two atoms of L are projective and the irreducibility of L is recovered.

10. Representations of Irreducible Proposition Systems

In this Section we shall sum up, without detailed proofs, a number of results about the analytical representation of an irreducible proposition system L .

We recall a few definitions. Let K be a division ring with an involutorial anti-automorphism $\lambda \mapsto \lambda^*$ (which means $(\lambda + \mu)^* = \lambda^* + \mu^*$, $(\lambda \mu)^* = \mu^* \lambda^*$, $\lambda^{**} = \lambda$) and let V be a vector space over K : a Hermitean form in V is a mapping f of $V \times V$ onto K satisfying the four conditions (where $x, x_i, y, y_i \in V$; $\lambda, \mu \in K$)

$$\begin{aligned} f(\lambda x_1 + \mu x_2, y) &= \lambda f(x_1, y) + \mu f(x_2, y), \\ f(x, \lambda y_1 + \mu y_2) &= \lambda^* f(x, y_1) + \mu^* f(x, y_2), \\ f(x, y) &= f(y, x)^*, \\ f(x, x) &= 0 \Rightarrow x = 0_V. \end{aligned}$$

Given a subspace W of V , put²⁴

$$W^\circ = \{x \in V : f(x, y) = 0 \text{ for every } y \in W\},$$

so that $W \cap W^\circ = 0_V$ and $W \subseteq W^{\circ\circ}$. The subspace W is said to be closed with respect to f if $W = W^{\circ\circ}$ (no

topology is involved here). The set $L_f(V)$ of all closed subspaces of V is a lattice in which the ordering relation is the set inclusion while the least upper bound and greatest upper bound are defined through

$$\begin{aligned} \bigvee_{i \in I} W_i &= \left(\sum_{i \in I} W_i \right)^{\circ\circ}, & W_i &\in L_f(V), \\ \bigwedge_{i \in I} W_i &= \bigcap_{i \in I} W_i, & W_i &\in L_f(V), \end{aligned}$$

where $\sum W_i$ stands for the algebraic sum of the subspaces W_i 's, ($\cap W_i$ is the set intersection). The lattice $L_f(V)$ is proved to meet all properties of proposition systems. If, for all subspaces W of V one has the decomposition $V = W + W^\circ$ (algebraic sum), then f is called Hilbertian.

The case of vector spaces V_n with finite dimension n is relevant; let us quote a result due to Birkhoff and Von Neumann^{1, 6, 24}. If V_n has dimension $n \geq 4$ and if the lattice $L(V_n)$ of all subspaces of V_n is orthocomplemented, then there exists an involutorial anti-automorphism $\lambda \mapsto \lambda^*$ of K and there exists an Hermitean form f in V_n such that for every $W \in L(V_n)$ the corresponding orthocomplement is given by

$$W^\perp = W^\circ = \{x \in V_n : f(x, y) = 0 \text{ for every } y \in W\}. \quad (10.1)$$

Moreover, the pair $(*, f)$ is unique in the following sense: if $\lambda \mapsto \lambda^*$ is another involutorial anti-automorphism of K and if \bar{f} is another Hermitean form (with respect to $\bar{*}$) satisfying (10.1), then there exists $\gamma \in K$ such that

$$\begin{aligned} \lambda^{\bar{*}} &= \gamma^{-1} \lambda^* \gamma, & \forall \lambda \in K, \\ \bar{f}(x, y) &= f(x, y) \gamma, & \forall x, y \in V_n. \end{aligned}$$

The main representation theorem of irreducible proposition systems now states⁴:

Let L be an irreducible proposition system of length³⁸ ≥ 4 , then there exist a division ring K with an involutorial antiautomorphism $\lambda \mapsto \lambda^*$ and a vector space V over K with an Hermitean and Hilbertian form f such that L is isomorphic to the lattice $L_f(V)$ of closed subspaces of V .

An elegant proof of this theorem may be found in the reference²⁴.

The isomorphism between the proposition lattice and $L_f(V)$ is proved by Varadarajan⁶ under a different (but equivalent) characterization of L : i.e.,

L is assumed to be a complete, atomic, orthocomplemented, weakly modular lattice for which

- (i) if $p \neq 0_L$ is the least upper bound of a finite number of atoms then $\mathcal{L}[0, p] = \{q \in L : q \leq p\}$ is a modular, irreducible lattice of finite length;
- (ii) if a is an atom of L and $p \neq 0_L, 1_L$ then there exist two atoms b, c such that $b < p, c < p^\perp, a < b \vee c$.

Gudder²⁶ has shown that part of the requirement (i) may be deduced from a more physical assumption (the so called minimal superposition principle).

At this stage it is natural to analyze which division rings K fulfil the requirements involved by the representation theorem. This problem has been studied for various number-fields^{39, 40}: it turns out that any algebraic extension or completion of p -adic fields and finite fields have to be excluded. Hence the only fields admitted must be derived, as algebraic extension or completion, from the unique archimedean, non p -adic valuation of the rational field (i.e. the usual absolute value). Thus one is left with real, complex or quaternion fields⁴¹. If K is the real field, the involutorial anti-automorphism $\lambda \mapsto \lambda^*$ is the identity, if it is the complex field, $\lambda \mapsto \lambda^*$ is the complex conjugation, if it is the quaternion field, $\lambda \mapsto \lambda^*$ is the so called canonical conjugation. A vector space V over one of these three fields, equipped with a Hermitean form, is called a pre-Hilbert space. It may be proved⁶ that any pre-Hilbert space with a Hilbertian form f is complete; summing up, one can state the following: let L be a lattice, L is isomorphic to the lattice of closed subspaces of a separable Hilbert space over the real, or complex, or quaternion field if and only if L is an irreducible proposition system in which every sequence of mutually orthogonal atoms is at most countable.

We emphasize that the restriction of K to reals, complex or quaternion fields is, in principle, quite independent from the choice of the number field to be used for the description of the geometry underlying the physical systems, say for the description of space-time. Anyhow it is likely that the description of the fundamental physical geometry by number systems different from the usual real field could induce significant modifications in the logical structure of states and propositions summarized in this paper.

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- ³⁸ The length of a lattice is the (uniquely determined) number of atoms, a_1, \dots, a_m such that $1_L = a_1 \vee \dots \vee a_m$ and $(a_1 \vee \dots \vee a_{i-1}) \wedge a_i = 0_L$ for $i = 2, \dots, m$.
- ³⁹ E. G. Beltrametti and G. Cassinelli, *Found. Phys.* **2**, 1 [1972].
- ⁴⁰ J. P. Eckmann and Ph. Ch. Zabey, *Helv. Phys. Acta* **42**, 420 [1969].
- ⁴¹ More precisely, quaternionic form a division ring.
- ^{*} Note in proofs:
... measurements and that all of them transform ...